

# FREE FIELD REALIZATION OF QUANTUM AFFINE SUPERALGEBRA $U_q(\widehat{sl}(N|1))$

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TAKEO KOJIMA

*Department of Mathematics and Physics, Graduate School of Science and Engineering,  
Yamagata University, Jonan 4-3-16, Yonezawa 992-8510, Japan*  
*kojima@yz.yamagata-u.ac.jp*

## Abstract

We construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  for an arbitrary level  $k \in \mathbb{C}$ .

# 1 Introduction

The free field approach [1] provides a powerful method to construct correlation functions of exactly solvable models. In this paper we construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  ( $N \geq 2$ ) for an arbitrary level  $k \in \mathbb{C}$ . The level parameter  $k$  plays an important role in representation theory. Free field realizations of an arbitrary level  $k \in \mathbb{C}$  are completely different from those of level  $k = 1$ . In the case of level  $k = 1$ , free field realizations [2, 3, 4] have been constructed for quantum affine algebra  $U_q(g)$  in many cases  $g = (ADE)^{(r)}$  [4, 7],  $(BC)^{(1)}$ ,  $G_2^{(1)}$  [5, 6, 8],  $\widehat{sl}(M|N)$ ,  $osp(2|2)^{(2)}$  [9, 10, 11]. In the case of an arbitrary level  $k \in \mathbb{C}$ , free field realizations [12, 13, 14], have not yet been studied well for quantum affine algebra  $U_q(g)$ . In the case of an arbitrary level  $k \in \mathbb{C}$ , free field realizations have been constructed only for  $U_q(\widehat{sl}(N))$  [16] and  $U_q(\widehat{sl}(2|1))$  [17]. The purpose of this paper is to construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  for an arbitrary level  $k \in \mathbb{C}$ . The representation theories of the superalgebra are much more complicated than non-superalgebra and have rich structures [18, 19, 20, 21].

This paper is organized as follows. In section 2 we review the Chevalley realization of the quantum superalgebra  $U_q(sl(N|1))$  [22] and the Drinfeld realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  [23]. In section 3 we review the Heisenberg realization of quantum superalgebra  $U_q(sl(N|1))$  [15] and construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  for an arbitrary level  $k \in \mathbb{C}$ . In appendix A we explain how to find the free field realization of affine  $U_q(\widehat{sl}(N|1))$  from the Heisenberg realization  $U_q(sl(N|1))$ . In appendix B we summarize some useful formulae.

## 2 Quantum Affine Superalgebra $U_q(\widehat{sl}(N|1))$

In this section we review the Chevalley realization of the quantum superalgebra  $U_q(sl(N|1))$  [22] and the Drinfeld realization of the quantum superalgebra  $U_q(\widehat{sl}(N|1))$  [23, 24] for  $N = 2, 3, 4, \dots$ . We fix a complex number  $q \neq 0, |q| < 1$ . In what follows we use

$$[x, y] = xy - yx, \quad (2.1)$$

$$\{x, y\} = xy + yx, \quad (2.2)$$

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (2.3)$$

## 2.1 Quantum Superalgebra $U_q(sl(N|1))$

Let us recall the definition of the quantum superalgebra  $U_q(sl(N|1))$  [22]. We set  $\nu_1 = \nu_2 = \dots = \nu_N = +, \nu_{N+1} = -$ . The Cartan matrix  $(A_{i,j})_{1 \leq i,j \leq N}$  of the Lie algebra  $sl(N|1)$  is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}. \quad (2.4)$$

The diagonal part is  $(A_{i,i})_{1 \leq i \leq N} = (\overbrace{2, \dots, 2}^{N-1}, 0)$ .

**Definition 2.1** [22] *The Chevalley generators of the quantum superalgebra  $U_q(sl(N|1))$  are*

$$h_i, e_i, f_i \quad (1 \leq i \leq N). \quad (2.5)$$

*Defining relations are*

$$[h_i, h_j] = 0, \quad (2.6)$$

$$[h_i, e_j] = A_{i,j}e_j, \quad (2.7)$$

$$[h_i, f_j] = -A_{i,j}f_j, \quad (2.8)$$

$$[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \quad \text{for } (i, j) \neq (N, N), \quad (2.9)$$

$$\{e_N, f_N\} = \frac{q^{h_N} - q^{-h_N}}{q - q^{-1}}, \quad (2.10)$$

*and the Serre relations*

$$e_i e_i e_j - (q + q^{-1})e_i e_j e_i + e_j e_i e_i = 0 \quad \text{for } |A_{i,j}| = 1, i \neq N, \quad (2.11)$$

$$f_i f_i f_j - (q + q^{-1})f_i f_j f_i + f_j f_i f_i = 0 \quad \text{for } |A_{i,j}| = 1, i \neq N. \quad (2.12)$$

## 2.2 Quantum Affine Superalgebra $U_q(\widehat{sl}(N|1))$

Let us recall the definition of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  [23]. The Cartan matrix  $(A_{i,j})_{0 \leq i,j \leq N}$  of the affine Lie algebra  $\widehat{sl}(N|1)$  is given by

$$A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}. \quad (2.13)$$

Here we should read the suffixes  $j$  of  $\nu_j \bmod (N+1)$ , i.e.  $\nu_0 = \nu_{N+1}$ . Here the diagonal part is

$$(A_{i,i})_{0 \leq i \leq N} = (0, \overbrace{2, \dots, 2}^{N-1}, 0).$$

**Definition 2.2** [23] The Drinfeld generators of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  are

$$x_{i,m}^{\pm}, h_{i,m}, c, \quad (1 \leq i \leq N, m \in \mathbb{Z}). \quad (2.14)$$

Defining relations are

$$c : \text{central}, [h_i, h_{j,m}] = 0, \quad (2.15)$$

$$[a_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m|} \delta_{m+n,0} \quad (m, n \neq 0), \quad (2.16)$$

$$[h_i, x_j^{\pm}(z)] = \pm A_{i,j} x_j^{\pm}(z), \quad (2.17)$$

$$[h_{i,m}, x_j^+(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m|} z^m x_j^+(z) \quad (m \neq 0), \quad (2.18)$$

$$[h_{i,m}, x_j^-(z)] = -\frac{[A_{i,j}m]_q}{m} z^m x_j^-(z) \quad (m \neq 0), \quad (2.19)$$

$$(z_1 - q^{\pm A_{i,j}} z_2) x_i^{\pm}(z_1) x_j^{\pm}(z_2) = (q^{\pm A_{j,i}} z_1 - z_2) x_j^{\pm}(z_2) x_i^{\pm}(z_1) \quad \text{for } |A_{i,j}| \neq 0, \quad (2.20)$$

$$x_i^{\pm}(z_1) x_j^{\pm}(z_2) = x_j^{\pm}(z_2) x_i^{\pm}(z_1) \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (N, N), \quad (2.21)$$

$$\{x_N^{\pm}(z_1), x_N^{\pm}(z_2)\} = 0, \quad (2.22)$$

$$[x_i^+(z_1), x_j^-(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1 / z_2) \psi_i^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_i^-(q^{-\frac{c}{2}} z_2) \right),$$

for  $(i, j) \neq (N, N),$  (2.23)

$$\{x_N^+(z_1), x_N^-(z_2)\} = \frac{1}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1 / z_2) \psi_N^+(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_N^-(q^{-\frac{c}{2}} z_2) \right), \quad (2.24)$$

$$\left( x_i^{\pm}(z_1) x_i^{\pm}(z_2) x_j^{\pm}(z) - (q + q^{-1}) x_i^{\pm}(z_1) x_j^{\pm}(z) x_i^{\pm}(z_2) + x_j^{\pm}(z) x_i^{\pm}(z_1) x_i^{\pm}(z_2) \right) \\ + (z_1 \leftrightarrow z_2) = 0 \quad \text{for } |A_{i,j}| = 1, i \neq N. \quad (2.25)$$

where we have used  $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$ . Here we have used the abbreviation  $h_i = h_{i,0}$ . We have set the generating function

$$x_j^{\pm}(z) = \sum_{m \in \mathbb{Z}} x_{j,m}^{\pm} z^{-m-1}, \quad (2.26)$$

$$\psi_i^+(q^{\frac{c}{2}} z) = q^{h_i} \exp \left( (q - q^{-1}) \sum_{m > 0} h_{i,m} z^{-m} \right), \quad (2.27)$$

$$\psi_i^-(q^{-\frac{c}{2}} z) = q^{-h_i} \exp \left( -(q - q^{-1}) \sum_{m > 0} h_{i,-m} z^m \right). \quad (2.28)$$

We changed the gauge of boson  $h_{i,m}$  from those of [23] and revised a misprint (2.22) in [23].

### 3 Free Field Realization

In this section we review the Heisenberg realization of  $U_q(sl(N|1))$  [17] and construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  for an arbitrary level  $k \in \mathbb{C}$ .

#### 3.1 Heisenberg Realization

Let us recall the Heisenberg realization of quantum superalgebra  $U_q(sl(N|1))$  [17]. We introduce the coordinates  $x_{i,j}$ , ( $1 \leq i < j \leq N+1$ ) by

$$x_{i,j} = \begin{cases} z_{i,j} & (1 \leq i < j \leq N), \\ \theta_{i,j} & (1 \leq i \leq N, j = N+1). \end{cases} \quad (3.1)$$

Here  $z_{i,j}$  are complex variables and  $\theta_{i,N+1}$  are the Grassmann odd variables that satisfy  $\theta_{i,N+1}\theta_{i,N+1} = 0$  and  $\theta_{i,N+1}\theta_{j,N+1} = -\theta_{j,N+1}\theta_{i,N+1}$ , ( $i \neq j$ ). We introduce the differential operators  $\vartheta_{i,j} = x_{i,j} \frac{\partial}{\partial x_{i,j}}$ , ( $1 \leq i < j \leq N+1$ ). We fix parameters  $\lambda_i \in \mathbb{C}$ , ( $1 \leq i \leq N$ ). We set the differential operators  $H_i, E_i, F_i$ , ( $1 \leq i \leq N$ ) by

$$H_i = \sum_{j=1}^N H_{i,j}, \quad E_i = \sum_{j=1}^i E_{i,j}, \quad F_i = \sum_{j=1}^N F_{i,j}. \quad (3.2)$$

Here we have set

$$H_{i,j} = \begin{cases} \nu_i \vartheta_{j,i} - \nu_{i+1} \vartheta_{j,i+1} & (1 \leq j \leq i-1), \\ \lambda_i - (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} & (j = i), \\ \nu_{i+1} \vartheta_{i+1,j+1} - \nu_i \vartheta_{i,j+1} & (i+1 \leq j \leq N), \end{cases} \quad (3.3)$$

$$E_{i,j} = \frac{x_{j,i}}{x_{j,i+1}} [\vartheta_{j,i+1}]_q q^{\sum_{l=1}^{j-1} (\nu_i \vartheta_{l,i} - \nu_{i+1} \vartheta_{l,i+1})}, \quad (3.4)$$

$$F_{i,j} = \begin{cases} \nu_i \frac{x_{j,i+1}}{x_{j,i}} [\vartheta_{j,i}]_q \times & (1 \leq j \leq i-1), \\ \times q^{\sum_{l=j+1}^{i-1} (\nu_{i+1} \vartheta_{l,i+1} - \nu_i \vartheta_{l,i}) - \lambda_i + (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} + \sum_{l=i+2}^{N+1} (\nu_i \vartheta_{i,l} - \nu_{i+1} \vartheta_{i+1,l})} & (j = i), \\ x_{i,i+1} \left[ \lambda_i - \nu_i \vartheta_{i,i+1} - \sum_{l=i+2}^{N+1} (\nu_i \vartheta_{i,l} - \nu_{i+1} \vartheta_{i+1,l}) \right]_q & (i+1 \leq j \leq N). \\ -\nu_{i+1} \frac{x_{i,j+1}}{x_{i+1,j+1}} [\vartheta_{i+1,j+1}]_q q^{\lambda_i + \sum_{l=j+1}^{N+1} (\nu_{i+1} \vartheta_{i+1,l} - \nu_i \vartheta_{i,l})} & \end{cases} \quad (3.5)$$

Here we read  $x_{i,i} = 1$  and, for Grassmann odd variables  $x_{i,j}$ , the expression  $\frac{1}{x_{i,j}}$  stands for the derivative  $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$ .

**Theorem 3.1** [17] *A Heisenberg realization of the quantum superalgebra  $U_q(sl(N|1))$  is given in the following way.*

$$h_i \rightarrow H_i, \quad (3.6)$$

$$e_i \rightarrow E_i, \quad (3.7)$$

$$f_i \rightarrow F_i. \quad (3.8)$$

In appendix A we explain how to find the free field realization of affine  $U_q(\widehat{sl}(N|1))$  from this Heisenberg realization  $U_q(sl(N|1))$ .

### 3.2 Boson

Let us fix the level  $c = k \in \mathbb{C}$ . Let us introduce the bosons and the zero-mode operators  $a_m^j, Q_a^j$  ( $m \in \mathbb{Z}, 1 \leq j \leq N$ ),  $b_m^{i,j}, Q_b^{i,j}, c_m^{i,j}, Q_c^{i,j}$  ( $m \in \mathbb{Z}, 1 \leq i < j \leq N+1$ ). The bosons  $a_m^i, b_m^{i,j}, c_m^{i,j}$ , ( $m \in \mathbb{Z}_{\neq 0}$ ) satisfy

$$[a_m^i, a_n^j] = \frac{[(k+N-1)m]_q [A_{i,j}m]_q}{m} \delta_{m+n,0}, \quad (3.9)$$

$$[b_m^{i,j}, b_n^{i',j'}] = -\nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad (3.10)$$

$$[c_m^{i,j}, c_n^{i',j'}] = \nu_i \nu_j \frac{[m]_q^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}. \quad (3.11)$$

The zero-mode operators  $a_0^i, Q_a^i, b_0^{i,j}, Q_b^{i,j}, c_0^{i,j}, Q_c^{i,j}$  satisfy

$$[a_0^i, Q_a^j] = (k+N-1)A_{i,j}, \quad (3.12)$$

$$[b_0^{i,j}, Q_b^{i',j'}] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, \quad (3.13)$$

$$[c_0^{i,j}, Q_c^{i',j'}] = \nu_i \nu_j \delta_{i,i'} \delta_{j,j'}. \quad (3.14)$$

and other commutators vanish. We impose the cocycle condition on the zero-mode operator  $Q_b^{i,j}$ , ( $1 \leq i < j \leq N+1$ ) by

$$[Q_b^{i,j}, Q_b^{i',j'}] = \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad \text{for } (i,j) \neq (i',j'). \quad (3.15)$$

We have the following (anti)commutation relations

$$\left[ \exp(Q_b^{i,j}), \exp(Q_b^{i',j'}) \right] = 0 \quad (1 \leq i < j \leq N, 1 \leq i' < j' \leq N), \quad (3.16)$$

$$\left\{ \exp(Q_b^{i,N+1}), \exp(Q_b^{j,N+1}) \right\} = 0 \quad (1 \leq i \neq j \leq N). \quad (3.17)$$

We use the following normal ordering symbol  $::$  as follows.

$$: b_m^{i,j} b_n^{i',j'} := \begin{cases} b_m^{i,j} b_n^{i',j'} & (m < 0), \\ b_n^{i',j'} b_m^{i,j} & (m > 0), \end{cases} \quad : a_m^i a_n^j := \begin{cases} a_m^i a_n^j & (m < 0), \\ a_n^j a_m^i & (m > 0), \end{cases} \quad (3.18)$$

$$: b_0^{i,j} Q_b^{i',j'} :=: Q_b^{i',j'} b_0^{i,j} := Q_b^{i',j'} b_0^{i,j}, \quad : a_0^i Q_a^j :=: Q_a^j a_0^i := Q_a^j a_0^i. \quad (3.19)$$

The above boson structure is the straightforward generalization of those in [17]. Note that  $(N-1)$  is the dual Coxeter number. In what follows we use  $\{a_m^j(1 \leq j \leq N), b_m^{i,j}, Q_b^{i,j}(1 \leq i < j \leq N+1), c_m^{i,j}, Q_c^{i,j}(1 \leq i < j \leq N)\}$  which is a subset of the above boson system. In what follows we use the abbreviations  $b^{i,j}(z), c^{i,j}(z), b_\pm^{i,j}(z), a_\pm^j(z)$ .

$$b^{i,j}(z) = - \sum_{m \neq 0} \frac{b_m^{i,j}}{[m]_q} z^{-m} + Q_b^{i,j} + b_0^{i,j} \log z, \quad (3.20)$$

$$c^{i,j}(z) = - \sum_{m \neq 0} \frac{c_m^{i,j}}{[m]_q} z^{-m} + Q_c^{i,j} + c_0^{i,j} \log z, \quad (3.21)$$

$$b_\pm^{i,j}(z) = \pm(q - q^{-1}) \sum_{\pm m > 0} b_m^{i,j} z^{-m} \pm b_0^{i,j} \log q, \quad (3.22)$$

$$a_\pm^j(z) = \pm(q - q^{-1}) \sum_{\pm m > 0} a_m^j z^{-m} \pm a_0^j \log q. \quad (3.23)$$

### 3.3 Free Field Realization

In this section we construct a free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  for an arbitrary level  $k$ . In [15], on the basis of the Heisenberg realization of the quantum algebra  $U_q(sl(N))$ , a free field realization of the quantum affine algebra  $U_q(\widehat{sl}(N))$  was obtained. Here we try to generalize it to the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$ . Detailed calculations of this trial are summarized in appendix A. We introduce the operators  $X_i^\pm(z), \Psi_i^\pm(z), (1 \leq i \leq N)$  on the Fock space as follows. For  $1 \leq i \leq N-1$  we introduce

$$X_i^+(z) = \frac{1}{(q - q^{-1})z} \sum_{j=1}^i (X_{i,2j-1}^+(z) - X_{i,2j}^+(z)), \quad (3.24)$$

$$X_N^+(z) = \sum_{j=1}^N X_{N,j}^+(z), \quad (3.25)$$

$$\begin{aligned} X_i^-(z) = & \frac{1}{(q - q^{-1})z} \left( \sum_{j=1}^{i-1} (X_{i,2j-1}^-(z) - X_{i,2j}^-(z)) + (X_{i,2i-1}^-(z) - X_{i,2i}^-(z)) \right. \\ & \left. - \sum_{j=i+1}^{N-1} (X_{i,2j-1}^-(z) - X_{i,2j}^-(z)) \right) + q^{k+N-1} X_{i,2N-1}^-(z), \end{aligned} \quad (3.26)$$

$$X_N^-(z) = \frac{1}{(q - q^{-1})z} \sum_{j=1}^N \left( -q^{j-1} X_{N,2j-1}^-(z) + q^{j-1} X_{N,2j}^-(z) \right). \quad (3.27)$$

$$\Psi_i^\pm(q^{\pm \frac{k}{2}} z) = \exp \left( a_\pm^i (q^{\pm \frac{k+N-1}{2}} z) + \sum_{l=1}^i (b_\pm^{l,i+1} (q^{\pm(l+k-1)} z) - b_\pm^{l,i} (q^{\pm(l+k)} z)) \right)$$

$$\begin{aligned}
& + \sum_{l=i+1}^N (b_{\pm}^{i,l}(q^{\pm(k+l)}z) - b_{\pm}^{i-1,l}(q^{\pm(k+l-1)}z)) \\
& + b_{\pm}^{i,N+1}(q^{\pm(k+N)}z) - b_{\pm}^{i+1,N+1}(q^{\pm(k+N-1)}z) \Big), \tag{3.28}
\end{aligned}$$

$$\Psi_N^{\pm}(q^{\pm \frac{k}{2}}z) = \exp \left( a_{\pm}^N(q^{\pm \frac{k+N-1}{2}}z) - \sum_{l=1}^{N-1} (b_{\pm}^{l,N}(q^{\pm(k+l)}z) + b_{\pm}^{l,N+1}(q^{\pm(k+l)}z)) \right). \tag{3.29}$$

Here we have used the auxiliary bosonic operators  $X_{i,j}^{\pm}(z)$  as follows.

For  $1 \leq i \leq N-1$  and  $1 \leq j \leq i$  we set

$$\begin{aligned}
X_{i,2j-1}^+(z) & = : \exp \left( (b+c)^{j,i}(q^{j-1}z) + b_+^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^jz) \right. \\
& \quad \left. + \sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z)) \right) :, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
X_{i,2j}^+(z) & = : \exp \left( (b+c)^{j,i}(q^{j-1}z) + b_-^{j,i+1}(q^{j-1}z) - (b+c)^{j,i+1}(q^{j-2}z) \right. \\
& \quad \left. + \sum_{l=1}^{j-1} (b_+^{l,i+1}(q^{l-1}z) - b_+^{l,i}(q^l z)) \right) :. \tag{3.31}
\end{aligned}$$

For  $1 \leq j \leq N$  we set

$$\begin{aligned}
X_{N,j}^+(z) & = : \exp \left( (b+c)^{j,N}(q^{j-1}z) + b^{j,N+1}(q^{j-1}z) \right. \\
& \quad \left. - \sum_{l=1}^{j-1} (b_+^{l,N+1}(q^l z) + b_+^{l,N}(q^l z)) \right) :. \tag{3.32}
\end{aligned}$$

For  $1 \leq i \leq N-1$  and  $1 \leq j \leq i-1$  we set

$$\begin{aligned}
X_{i,2j-1}^-(z) & = : \exp \left( a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) \right. \\
& \quad - b_-^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j+1}z) \\
& \quad + \sum_{l=j+1}^i (b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) \\
& \quad + \sum_{l=i+1}^N (b_-^{l,i}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) \\
& \quad \left. + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z) \right) :, \tag{3.33} \\
X_{i,2j}^-(z) & = : \exp \left( a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{j,i+1}(q^{-k-j}z) \right. \\
& \quad - b_+^{j,i}(q^{-k-j}z) - (b+c)^{j,i}(q^{-k-j-1}z) \\
& \quad + \sum_{l=j+1}^i (b_-^{l,i+1}(q^{-k-l+1}z) - b_-^{l,i}(q^{-k-l}z)) \\
& \quad \left. + \sum_{l=i+1}^N (b_-^{l,i}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) \right) :
\end{aligned}$$



$$+b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z) \Big) : . \quad (3.34)$$

For  $1 \leq i \leq N-1$  we set

$$\begin{aligned} X_{i,2i-1}^-(z) &= : \exp \left( a_-^i(q^{-\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{-k-i}z) \right. \\ &\quad + \sum_{l=i+1}^N (b_-^{i,l}(q^{-k-l}z) - b_-^{i+1,l}(q^{-k-l+1}z)) \\ &\quad \left. + b_-^{i,N+1}(q^{-k-N}z) - b_-^{i+1,N+1}(q^{-k-N+1}z) \right) :, \end{aligned} \quad (3.35)$$

$$\begin{aligned} X_{i,2i}^-(z) &= : \exp \left( a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,i+1}(q^{k+i}z) \right. \\ &\quad + \sum_{l=i+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) \\ &\quad \left. + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z) \right) : . \end{aligned} \quad (3.36)$$

For  $1 \leq i \leq N-1$  and  $i+1 \leq j \leq N-1$  we set

$$\begin{aligned} X_{i,2j-1}^-(z) &= : \exp \left( a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,j+1}(q^{k+j}z) \right. \\ &\quad + b_+^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j+1}z) \\ &\quad + \sum_{l=j+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) \\ &\quad \left. + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z) \right) :, \end{aligned} \quad (3.37)$$

$$\begin{aligned} X_{i,2j}^-(z) &= : \exp \left( a_+^i(q^{\frac{k+N-1}{2}}z) + (b+c)^{i,j+1}(q^{k+j}z) \right. \\ &\quad + b_-^{i+1,j+1}(q^{k+j}z) - (b+c)^{i+1,j+1}(q^{k+j-1}z) \\ &\quad + \sum_{l=j+1}^N (b_+^{i,l}(q^{k+l}z) - b_+^{i+1,l}(q^{k+l-1}z)) \\ &\quad \left. + b_+^{i,N+1}(q^{k+N}z) - b_+^{i+1,N+1}(q^{k+N-1}z) \right) : . \end{aligned} \quad (3.38)$$

For  $1 \leq i \leq N-1$  we set

$$\begin{aligned} X_{i,2N-1}^-(z) &= : \exp \left( a_+^i(q^{\frac{k+N-1}{2}}z) - b^{i,N+1}(q^{k+N-1}z) \right. \\ &\quad \left. - b_+^{i+1,N+1}(q^{k+N-1}z) + b^{i+1,N+1}(q^{k+N}z) \right) : . \end{aligned} \quad (3.39)$$

For  $1 \leq j \leq N-1$  we set

$$\begin{aligned} X_{N,2j-1}^-(z) &= : \exp \left( a_-^N(q^{-\frac{k+N-1}{2}}z) - b_-^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j+1}z) \right. \\ &\quad \left. - b_-^{j,N+1}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j+1}z) \right) : . \end{aligned}$$

$$- \sum_{l=j+1}^{N-1} (b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z)) \Big) :, \quad (3.40)$$

$$\begin{aligned} X_{N,2j}^-(z) = & : \exp \left( a_-^N(q^{-\frac{k+N-1}{2}}z) - b_+^{j,N}(q^{-k-j}z) - (b+c)^{j,N}(q^{-k-j-1}z) \right. \\ & \left. - b_+^{j,N+1}(q^{-k-j}z) - b^{j,N+1}(q^{-k-j-1}z) \right. \\ & \left. - \sum_{l=j+1}^{N-1} (b_-^{l,N}(q^{-k-l}z) + b_-^{l,N+1}(q^{-k-l}z)) \right) :, \end{aligned} \quad (3.41)$$

$$X_{N,2N-1}^-(z) = : \exp \left( a_-^N(q^{-\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{-k-N+1}z) \right) :, \quad (3.42)$$

$$X_{N,2N}^-(z) = : \exp \left( a_+^N(q^{\frac{k+N-1}{2}}z) - b^{N,N+1}(q^{k+N-1}z) \right) :. \quad (3.43)$$

Now we have introduced the bosonic operators  $X_i^\pm(z)$  and  $\Psi_i^\pm(z)$ .

The following is **main result** of this paper.

**Theorem 3.2** *A free field realization of the quantum affine superalgebra  $U_q(\widehat{sl}(N|1))$  is given in the following way.*

$$c \mapsto k \quad (3.44)$$

$$x_i^\pm(z) \mapsto X_i^\pm(z) \quad (3.45)$$

$$\psi_i^\pm(z) \mapsto \Psi_i^\pm(z). \quad (3.46)$$

In other words, the above map gives a homomorphism from  $U_q(\widehat{sl}(N|1))$  to the bosonic operator.

Very explicitly the relation (3.46) is written as

$$\begin{aligned} h_{i,m} \mapsto & q^{-\frac{k+N-1}{2}|m|} a_m^i + \sum_{l=1}^i (q^{-(k+l-1)|m|} b_m^{l,i+1} - q^{-(k+l)|m|} b_m^{l,i}) \\ & + \sum_{l=i+1}^N (q^{-(k+l)|m|} b_m^{i,l} - q^{-(k+l-1)|m|} b_m^{i+1,l}) \\ & + q^{-(k+N)|m|} b_m^{i,N+1} - q^{-(k+N-1)|m|} b_m^{i+1,N+1} \quad (1 \leq i \leq N-1), \end{aligned} \quad (3.47)$$

$$h_{N,m} \mapsto q^{-\frac{k+N-1}{2}|m|} a_m^N - \sum_{l=1}^{N-1} (q^{-(k+l)|m|} b_m^{l,N} + q^{-(k+l)|m|} b_m^{l,N+1}). \quad (3.48)$$

We give some comments on this realization. Upon the specialization  $N = 2$ , this free field realization reproduces the result for  $U_q(\widehat{sl}(2|1))$  in [17]. The structure of non-superalgebra  $U_q(\widehat{sl}(N))$  exists inside the superalgebra  $U_q(\widehat{sl}(N|1))$ . Hence the free field realizations of the currents  $X_i^\pm(z)$  ( $i \neq N$ ) for  $U_q(\widehat{sl}(N|1))$  are quite similar as those for  $U_q(\widehat{sl}(N))$ . The free field realizations of the fermionic operators  $X_{N,j}^+(z)$ ,  $X_{N,2j-1}^-(z)$ ,  $X_{N,2j}^-(z)$  and  $X_{j,2N-1}^-(z)$  of

$U_q(\widehat{sl}(N|1))$  are completely different from those of  $U_q(\widehat{sl}(N))$ . The free field realization of this paper is not irreducible representation. We have to construct screening currents that commute with the currents  $X_j^\pm(z)$  in order to get an irreducible representation [26, 27, 28]. We would like report this subject in the future publication. Applying the dressing method developed in [25] to this theorem, we have a free field realization of the elliptic algebra  $U_{q,p}(\widehat{sl}(N|1))$ .

*Proof of Theorem.* Direct calculations of the normal orderings show this theorem. The normal orderings of bosonic operators  $X_{i,j}^\pm(z)$  ( $i \neq N, j \neq 2N-1$ ) of the superalgebra  $U_q(\widehat{sl}(N|1))$  are exactly the same as those of the non-superalgebra  $U_q(\widehat{sl}(N))$ . Hence the proof of the relations for the bosonic operators  $X_i^\pm(z)$  ( $i \neq N$ ) is exactly the same as those of  $U_q(\widehat{sl}(N))$ . Let us focus our attention on the fermionic operators  $X_N^\pm(z)$  that is new for the superalgebra. We show the following relations for the fermionic operators  $X_N^\pm(z)$ .

$$\begin{aligned} & \{X_N^+(z_1), X_N^-(z_2)\} \\ &= \frac{1}{(q - q^{-1})z_1 z_2} \left( \delta(q^k z_2/z_1) \Psi_N^+(q^{\frac{k}{2}} z_2) - \delta(q^{-k} z_2/z_1) \Psi_N^-(q^{-\frac{k}{2}} z_2) \right), \end{aligned} \quad (3.49)$$

and

$$[X_N^+(z_1), X_j^-(z_2)] = 0 \quad \text{for } 1 \leq j \leq N-1. \quad (3.50)$$

First, let us show (3.49). Using the relation (B.5) in appendix B, we have

$$\begin{aligned} & \{X_N^+(z_1), X_N^-(z_2)\} \\ &= \frac{1}{(q - q^{-1})z_2} \sum_{j=1}^N q^{j-1} \left( -\{X_{N,j}^+(z_1), X_{N,2j-1}^-(z_2)\} + \{X_{N,j}^+(z_1), X_{N,2j}^-(z_2)\} \right). \end{aligned}$$

Using the relations (B.1), (B.2), (B.3) and (B.4) in appendix B, we have

$$\begin{aligned} & \{X_N^+(z_1), X_N^-(z_2)\} = \frac{1}{(q - q^{-1})z_1 z_2} \left( \delta(q^k z_2/z_1) \Psi_N^+(q^{\frac{k}{2}} z_2) - \delta(q^{-k} z_2/z_1) \Psi_N^-(q^{-\frac{k}{2}} z_2) \right) \\ & + \frac{1}{(q - q^{-1})z_1 z_2} \exp \left( a_-^N (q^{-\frac{k+N-1}{2}} z_2) \right) \times \\ & \left\{ \sum_{j=1}^{N-1} \delta \left( \frac{q^{-k-2j} z_2}{z_1} \right) : \exp \left( - \sum_{l=1}^j (b_+^{l,N} (q^l z_1) + b_+^{l,N+1} (q^l z_1)) - \sum_{l=j+1}^{N-1} (b_-^{l,N} (q^{-k-l} z_2) + b_+^{l,N+1} (q^{-k-l} z_2)) \right) : \right. \\ & \left. - \sum_{j=2}^N \delta \left( \frac{q^{-k-2j+2} z_2}{z_1} \right) : \exp \left( - \sum_{l=1}^{j-1} (b_+^{l,N} (q^l z_1) + b_+^{l,N+1} (q^l z_1)) - \sum_{l=j}^{N-1} (b_-^{l,N} (q^{-k-l} z_2) + b_+^{l,N+1} (q^{-k-l} z_2)) \right) : \right\}. \end{aligned}$$

Making the transformation  $j \rightarrow j-1$  in the first sum  $\sum_{j=1}^{N-1} \delta(q^{-k-2j} z_2/z_1)$ , we see cancellations.

We have the relation (3.49).

Next, let us show (3.50). Using the relation (B.9) in appendix B, we have the following for  $1 \leq j \leq N-2$ .

$$\begin{aligned} & [X_N^+(z_1), X_j^-(z_2)] \\ &= \frac{-1}{(q - q^{-1})z_2} [X_{N,j}^+(z_1), X_{j,2N-3}^-(z_2)] + q^{k+N-1} [X_{N,j+1}^+(z_1), X_{j,2N-1}^-(z_2)]. \end{aligned}$$

Using the relations (B.6), (B.8) in appendix B, we have

$$\begin{aligned} & [X_N^+(z_1), X_j^-(z_2)] = \delta \left( \frac{q^{k+N-j}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k+N-j}}{z_1} \right) \\ & \times : \exp \left( a_+^j (q^{\frac{k+N-1}{2}} z_2) - b_+^{j+1,N+1} (q^{k+N-1} z_2) + b^{j+1,N+1} (q^{k+N} z_2) + (b+c)^{j,N} (q^{k+N-1} z_2) \right. \\ & \left. - \sum_{l=1}^{j-1} (b_+^{l,N} (q^{k+N-j+l} z_2) + b_+^{l,N+1} (q^{k+N-j+l} z_2)) \right) : . \end{aligned}$$

From the relation  $\left( -\frac{1}{z_2} + \frac{q^{k+N-j}}{z_1} \right) \delta \left( \frac{q^{k+N-j}z_2}{z_1} \right) = 0$ , we have

$$[X_N^+(z_1), X_j^-(z_2)] = 0 \quad \text{for } 1 \leq j \leq N-2.$$

From the relation (B.9) in appendix B, we have

$$\begin{aligned} & [X_N^+(z_1), X_{N-1}^-(z_2)] \\ &= \frac{-1}{(q - q^{-1})z_2} [X_{N,N-1}^+(z_1), X_{N-1,2N-2}^-(z_2)] + q^{k+N-1} [X_{N,N}^+(z_1), X_{N-1,2N-1}^-(z_2)]. \end{aligned}$$

Using the relations (B.7), (B.8) and the relation  $\delta \left( \frac{q^{k-1}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k-1}}{z_1} \right) = 0$ , we have

$$\begin{aligned} & [X_N^+(z_1), X_{N-1}^-(z_2)] = \delta \left( \frac{q^{k-1}z_2}{z_1} \right) \left( -\frac{1}{z_2} + \frac{q^{k-1}}{z_1} \right) \\ & \times : \exp \left( a_+^{N-1} (q^{\frac{k+N-1}{2}} z_2) - b_+^{N,N+1} (q^{k+N-1} z_2) + b^{N,N+1} (q^{k+N} z_2) + (b+c)^{N-1,N} (q^{k+N-1} z_2) \right. \\ & \left. - \sum_{l=1}^{N-2} (b_+^{l,N} (q^{k+l+1} z_2) + b_+^{l,N+1} (q^{k+l+1} z_2)) \right) := 0. \end{aligned}$$

We have shown the relation (3.50). Q.E.D.

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## A Replacement

In this appendix we explain how to find the free field realization of affine  $U_q(\widehat{sl}(N|1))$  from the Heisenberg realization of  $U_q(sl(N|1))$ .

### A.1 Basic Operator

We would like to explain the role of the basic operators

$$: \exp(\pm b^{i,N+1}(z)) : , \quad : \exp\left(b_{\pm}^{i,j}(z) \pm (b+c)^{i,j}(q^{\mp 1}z)\right) :, \quad (\text{A.1})$$

which have been used for  $U_q(\widehat{sl}(2|1))$  [17] and  $U_q(\widehat{sl}(2))$  [14], respectively. The basic operators  $: \exp(\pm b^{i,N+1}(z)) :$  ( $1 \leq i \leq N$ ) satisfy the fermionic relation

$$\{ : \exp(b^{i,N+1}(z_1)) : , : \exp(-b^{i,N+1}(z_2)) : \} = \frac{1}{z_1} \delta(z_2/z_1). \quad (\text{A.2})$$

The basic operators  $: \exp(\pm b^{i,N+1}(z)) :$  create the delta-function  $\delta(z)$  and play important roles in constructions of the fermionic operators  $X_N^{\pm}(z)$  that satisfy

$$\{X_N^+(z_1), X_N^-(z_2)\} = \frac{1}{(q-q^{-1})z_1z_2} \left( \delta(q^k z_2/z_1) \Psi_N^+(q^{\frac{k}{2}} z_2) - \delta(q^{-k} z_2/z_1) \Psi_N^-(q^{-\frac{k}{2}} z_2) \right).$$

The basic operators  $: \exp\left(b_{\pm}^{i,j}(z) \pm (b+c)^{i,j}(q^{\mp 1}z)\right) :$  ( $1 \leq i < j \leq N$ ) satisfy the bosonic relations

$$\begin{aligned} & \left[ : \exp\left(b_+^{i,j}(z_1) - (b+c)^{i,j}(qz_1)\right) : , : \exp\left(b_+^{i,j}(z_2) + (b+c)^{i,j}(q^{-1}z_2)\right) : \right] \\ &= (q^{-1} - q) \delta(q^{-2} z_2/z_1) : \exp\left(b_+^{i,j}(z_1) + b_+^{i,j}(z_2)\right) :, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} & \left[ : \exp\left(b_-^{i,j}(z_1) - (b+c)^{i,j}(q^{-1}z_1)\right) : , : \exp\left(b_-^{i,j}(z_2) + (b+c)^{i,j}(qz_2)\right) : \right] \\ &= (q - q^{-1}) \delta(q^2 z_2/z_1) : \exp\left(b_-^{i,j}(z_1) + b_-^{i,j}(z_2)\right) : . \end{aligned} \quad (\text{A.4})$$

The basic operators  $: \exp\left(b_{\pm}^{i,j}(z) \pm (b+c)^{i,j}(q^{\mp 1}z)\right) :$  create the delta-function  $\delta(z)$  and play important roles in constructions of the bosonic operators  $X_i^{\pm}(z)$  ( $i \neq N$ ) that satisfy

$$[X_i^+(z_1), X_j^-(z_2)] = \frac{\delta_{i,j}}{(q-q^{-1})z_1z_2} \left( \delta(q^k z_2/z_1) \Psi_i^+(q^{\frac{k}{2}} z_2) - \delta(q^{-k} z_2/z_1) \Psi_i^-(q^{-\frac{k}{2}} z_2) \right).$$

Multiplying and adding proper operators to these basic operators (A.1), we construct the free field realization. For this purpose, the following replacement from the Heisenberg realization of  $U_q(\widehat{sl}(N|1))$  to the free field realization of the affine  $U_q(\widehat{sl}(N|1))$  gives useful information.

## A.2 Replacement

In this appendix we explain how to find the free field realization of the affine superalgebra  $U_q(\widehat{sl}(N|1))$  from the Heisenberg realization of  $U_q(\widehat{sl}(N|1))$ . We make the following replacement with suitable argument.

$$\vartheta_{i,j} \rightarrow -b_{\pm}^{i,j}(z)/\log q \quad (1 \leq i < j \leq N+1), \quad (\text{A.5})$$

$$[\vartheta_{i,j}]_q \rightarrow \begin{cases} \frac{\exp(\pm b_{+}^{i,j}(z)) - \exp(\pm b_{-}^{i,j}(z))}{(q - q^{-1})z} & (j \neq N+1), \\ 1 & (j = N+1). \end{cases} \quad (\text{A.6})$$

$$x_{i,j} \rightarrow \begin{cases} : \exp((b+c)^{i,j}(z)) : & (j \neq N+1), \\ : \exp(-b^{i,j}(z)) : \text{ or } : \exp(-b_{\pm}^{i,j}(q^{\pm 1}z) - b^{i,j}(z)) : & (j = N+1). \end{cases} \quad (\text{A.7})$$

$$\lambda_i \rightarrow a_{\pm}^i(z)/\log q \quad (1 \leq i \leq N), \quad (\text{A.8})$$

$$[\lambda_i]_q \rightarrow \frac{\exp(\pm a_{+}^i(z)) - \exp(\pm a_{-}^i(z))}{(q - q^{-1})z} \quad (1 \leq i \leq N). \quad (\text{A.9})$$

Taking the basic operators (A.1) into account, we gave this rule of the replacement.

From the above replacement,  $H_i$  of the Heisenberg realization (3.2) is replaced as following.

$$q^{H_i} \rightarrow \begin{cases} \exp\left(a_{\pm}^i(z) + \sum_{l=1}^i (b_{\pm}^{l,i+1}(z) - b_{\pm}^{l,i}(z)) + \sum_{l=i+1}^N (b_{\pm}^{i,l}(z) - b_{\pm}^{i+1,l}(z))\right) & (1 \leq i \leq N-1), \\ \exp\left(a_{\pm}^N(z) - \sum_{l=1}^{N-1} (b_{\pm}^{l,N}(z) + b_{\pm}^{l,N+1}(z))\right) & (i = N). \end{cases} \quad (\text{A.10})$$

There exist small gaps between the above operators (A.10) and the free field realizations  $\Psi_i^{\pm}(z)$  (3.28), (3.29). In order to make the operators (A.10) satisfy the defining relations of  $U_q(\widehat{sl}(N|1))$ , we have to impose  $q$ -shift to variable  $z$  of the operators  $a_{\pm}^i(z)$ ,  $b_{\pm}^{i,j}(z)$ . For instance, we have to replace  $a_{\pm}^i(z) \rightarrow a_{\pm}^i(q^{\pm \frac{k+N-1}{2}}z)$ . Bridging the gap by the  $q$ -shift, we have the free field realizations  $\Psi_i^{\pm}(q^{\pm \frac{k}{2}}z)$  (3.28), (3.29) from  $q^{H_i}$ .

$$q^{H_i} \rightarrow \Psi_i^{\pm}(q^{\pm \frac{k}{2}}z) \quad (1 \leq i \leq N). \quad (\text{A.11})$$

The structure of non-superalgebra  $U_q(\widehat{sl}(N))$  exists inside the superalgebra  $U_q(\widehat{sl}(N|1))$ . Hence the free field realizations of the currents  $X_i^{\pm}(z)$  ( $i \neq N$ ) for  $U_q(\widehat{sl}(N|1))$  are quite similar

as those for  $U_q(\widehat{sl}(N))$ . Let us focus our attention on the fermionic operators  $X_N^\pm(z)$  that is new for the superalgebra. Let us consider  $E_N = \sum_{j=1}^N E_{N,j}$  of the Heisenberg realization (3.2). From the above replacement, we have

$$E_{N,j} \rightarrow : \exp \left( (b+c)^{j,N}(z) + b^{j,N+1}(z) - \sum_{l=1}^{j-1} (b_+^{l,N}(z) + b_+^{l,N+1}(z)) \right) :. \quad (\text{A.12})$$

There exists an ambiguity of the replacement of  $x_{j,N+1}$  in (A.7). Here we have chose the replacement  $x_{j,N+1} \rightarrow: \exp(-b^{j,N+1}(z)) : (1 \leq j \leq N)$ . Imposing proper  $q$ -shift to the variable  $z$  of the operators  $(b+c)^{j,N}(z)$ ,  $b^{j,N+1}(z)$ ,  $b_\pm^{i,j}(z)$ , we have the free field realizations  $X_{N,j}^+(z)$  in (3.32).

$$E_{N,j} \rightarrow X_{N,j}^+(z) \quad (1 \leq j \leq N). \quad (\text{A.13})$$

Let us consider  $F_N = \sum_{j=1}^N F_{N,j}$  of the Heisenberg realization (3.2). From the above replacement we have

$$F_{N,j} \rightarrow \frac{1}{(q-q^{-1})z} \times \begin{cases} : \exp \left( -a_-^N(z) - b^{j,N+1}(z) - (b+c)^{j,N}(z) + \sum_{l=j+1}^{N-1} (b_-^{l,N+1}(z) - b_-^{l,N}(z)) \right) \\ \times \left( \exp \left( -b_+^{j,N}(z) - b_+^{j,N+1}(z) \right) - \exp \left( -b_-^{j,N}(z) - b_-^{j,N+1}(z) \right) \right) : & (j \neq N), \\ : \exp(-b^{N,N+1}(z)) (\exp(a_+^N(z)) - \exp(a_-^N(z))) : & (j = N). \end{cases} \quad (\text{A.14})$$

There exists an ambiguity of the replacement of  $x_{j,N+1}$  in (A.7). Here we have chose the replacement  $x_{j,N+1} \rightarrow: \exp(b_\pm^{j,N+1}(q^{\mp 1}z) - b^{j,N+1}(z)) : (1 \leq j \leq N-1)$  and  $x_{N,N+1} \rightarrow: \exp(-b^{N,N+1}(z)) :. Imposing proper  $q$ -shift to the variable  $z$  of the operators  $(b+c)^{j,N}(z)$ ,  $b^{j,N+1}(z)$ ,  $b_\pm^{i,j}(z)$ ,  $a_-^N(z)$ , we have the free field realizations  $X_{N,2j-1}^-(z)$ ,  $X_{N,2j}^-(z)$  in (3.40), (3.41), (3.42) and (3.43).$

$$F_{N,j} \rightarrow \frac{-1}{(q-q^{-1})z} (X_{N,2j-1}^-(z) - X_{N,2j}^-(z)) \quad (1 \leq j \leq N). \quad (\text{A.15})$$

Replacements for bosonic operators  $X_j^\pm(z)$ , ( $j \neq N$ ) have already appeared in  $U_q(\widehat{sl}(N))$  [16]. We explained details of the replacement for the fermionic operator  $X_N^\pm(z)$ , which is new for the superalgebra.

## B Normal Orderings

In this appendix we summarize useful relations.

For  $1 \leq j \leq N$  we have

$$\{X_{N,j}^+(z_1), X_{N,2j-1}^-(z_2)\} = \frac{1}{q^{j-1}z_1} \delta(q^{-k-2j+2}z_2/z_1)$$

$$\begin{aligned}
\times & : \exp \left( a_-^N (q^{-\frac{k+N-1}{2}} z_2) - \sum_{l=1}^{j-1} (b_+^{l,N} (q^{-k-2j+l+2} z_2) + b_+^{l,N+1} (q^{-k-2j+l+2} z_2)) \right. \\
& \left. - \sum_{l=j}^{N-1} (b_-^{l,N} (q^{-k-l} z_2) + b_-^{l,N+1} (q^{-k-l} z_2)) \right) : .
\end{aligned} \tag{B.1}$$

Especially for  $j = 1$  we have

$$\{X_{N,1}^+(z_1), X_{N,1}^-(z_2)\} = \frac{1}{z_1} \delta(q^{-k} z_2 / z_1) \Psi_N^-(q^{-\frac{k}{2}} z_2). \tag{B.2}$$

For  $1 \leq j \leq N-1$  we have

$$\begin{aligned}
\{X_{N,j}^+(z_1), X_{N,2j}^-(z_2)\} &= \frac{1}{q^{j-1} z_1} \delta(q^{-k-2j} z_2 / z_1) \\
\times & : \exp \left( a_-^N (q^{-\frac{k+N-1}{2}} z_2) - \sum_{l=1}^j (b_+^{l,N} (q^{-k-2j+l} z_2) + b_+^{l,N+1} (q^{-k-2j+l} z_2)) \right. \\
& \left. - \sum_{l=j+1}^{N-1} (b_-^{l,N} (q^{-k-l} z_2) + b_-^{l,N+1} (q^{-k-l} z_2)) \right) :,
\end{aligned} \tag{B.3}$$

$$\{X_{N,N}^+(z_1), X_{N,2N}^-(z_2)\} = \frac{1}{q^{N-1} z_1} \delta(q^k z_2 / z_1) \Psi_N^+(q^{\frac{k}{2}} z_2). \tag{B.4}$$

Other anti-commutators relations  $\{X_{N,i}^+(z_1), X_{N,j}^-(z_2)\}$  vanish.

$$\{X_{N,i}^+(z_1), X_{N,j}^-(z_2)\} = 0 \quad \text{for } j \neq 2i-1, 2i. \tag{B.5}$$

For  $1 \leq j \leq N-2$  we have

$$\begin{aligned}
[X_{N,j+1}^+(z_1), X_{j,2N-3}^-(z_2)] &= (q - q^{-1}) \delta(q^{k+N-j} z_2 / z_1) \\
\times & : \exp \left( a_+^j (q^{\frac{k+N-1}{2}} z_2) - b_+^{j+1,N+1} (q^{k+N-1} z_2) \right. \\
& + b_+^{j+1,N+1} (q^{k+N} z_2) + (b+c)^{j,N} (q^{k+N-1} z_2) \\
& \left. - \sum_{l=1}^{j-1} (b_+^{l,N} (q^{k+N-j+l} z_2) + b_+^{l,N+1} (q^{k+N-j+l} z_2)) \right) : .
\end{aligned} \tag{B.6}$$

We have

$$\begin{aligned}
[X_{N,N}^+(z_1), X_{N-1,2N-2}^-(z_2)] &= (q - q^{-1}) \delta(q^{k+1} z_2 / z_1) \\
\times & : \exp \left( a_+^{N-1} (q^{\frac{k+N-1}{2}} z_2) - b_+^{N,N+1} (q^{k+N-1} z_2) \right. \\
& + b_+^{N,N+1} (q^{k+N} z_2) + (b+c)^{N-1,N} (q^{k+N-1} z_2) \\
& \left. - \sum_{l=1}^{N-2} (b_+^{l,N} (q^{k+l+1} z_2) + b_+^{l,N+1} (q^{k+l+1} z_2)) \right) : .
\end{aligned} \tag{B.7}$$



For  $1 \leq j \leq N-1$  we have

$$\begin{aligned}
[X_{N,j}^+(z_1), X_{j,2N-1}^-(z_2)] &= \frac{1}{q^{j-1}z_1} \delta(q^{k+N-j}z_2/z_1) \\
&\times : \exp \left( a_+^j(q^{\frac{k+N-1}{2}}z_2) - b_+^{j+1,N+1}(q^{k+N-1}z_2) \right. \\
&\quad + b_+^{j+1,N+1}(q^{k+N}z_2) + (b+c)^{j,N}(q^{k+N-1}z_2) \\
&\quad \left. - \sum_{l=1}^{j-1} (b_+^{l,N}(q^{k+N-j+l}z_2) + b_+^{l,N+1}(q^{k+N-j+l}z_2)) \right) : . \quad (B.8)
\end{aligned}$$

Other commutation relations  $[X_{N,i}^+(z_1), X_{l,j}^-(z_2)]$  vanish.

$$[X_{N,i}^+(z_1), X_{j,l}^-(z_2)] = 0 \quad \text{for } (i, j, l) \neq \begin{cases} (j, j, 2N-1) & (1 \leq j \leq N-1), \\ (j+1, j, 2N-3) & (1 \leq j \leq N-2), \\ (N, N-1, 2N-2) & . \end{cases} \quad (B.9)$$

For  $1 \leq i \leq N-1$  we have

$$[X_{i,2}^+(z_1), X_{i,1}^-(z_2)] = (q - q^{-1}) \delta(q^{-k}z_2/z_1) \Psi_i^-(q^{-\frac{k}{2}}z_2), \quad (B.10)$$

$$[X_{i,2i-1}^+(z_1), X_{i,2i}^-(z_2)] = -(q - q^{-1}) \delta(q^kz_2/z_1) \Psi_i^+(q^{\frac{k}{2}}z_2). \quad (B.11)$$

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